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Impulsive Stabilization of a Class of Stochastic Functional Differential Equations with Time Delays

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***Abstract*—This paper investigates the pth moment globally uniformly exponential stability of a class of impulsive stabilization of stochastic delay differential equations,and the pth moment exponential stability criteria is established by using theLyapunov–Razumikhin method.**

***Keywords*—Stochastic delay differential equations; Exponential stability ; Lyapunov-Razumikhin method ; Impulsive control**

I. INTRODUCTION

In recent years, impulsive control and stabilization has been shown to be a powerful tool in the theory and applications of nonlinear dynamical systems.,such as control systems[1,2] . In particular, special attention has been focused on exponential stability of delay differential equations because it has played an important role in many areas[3-6]. However, to the best of the authors' knowledge, there are few studies on impulsive stabilization of stochastic delay systems.In[7], P.Cheng, F.Q.Deng proved several criteria on global exponential stability of impulsive stochastic functional differential systems by utilizing Lyapunov function methods combined with Razumikhin techniques. The result shows that impulses do contribute to global exponential stability of dynamical systems with any time delays even if they are unstable.In[8], J. Liu, X. Liu, W.C. Xie proved both moment and almost sure exponential stability criteria on impulsive stabilization of stochastic delay differential equations can be established by using the Lyapunov–Razumikhin method.

This paper formulates a simple impulsive stabilization of stochastic delay differential equations which would unstable,the aim of this work is to show this system must be stable if given several criteria on it.

II. PRELIMINARIES

Consider the following impulsive stabilization of stochastic delay differential equations:

$$\begin{cases} dx(t)=[A(t)x(t)+B(t)x(t-r(t))]dt+[Cx(t)+Dx(t-r(t))]dW(t), t\neq t_k, t>t_0 \\ \Delta x(t)=Ix(t^-), t=t_k, \\ x_{t_0}=\xi, \end{cases} \tag{2.1}$$

where $A(t)$, $B(t)$ are all $n\times n$ function matrices which are continuous on $[t_0,\infty)$, C , D , I are all $n\times n$ matrices and W is a one-dimensional standard Wiener process. A single time-varying delay is given by $r(t)$, which is continuous on $[t_0,\infty)$ and satisfies $0\le r(t)\le r$, for some constant $r>0$. The initial data is $\xi\in L^b_{f_{t_0}}$ .In addition, it is assumed that $A(t)x(t)+B(t)x(t-r(t))=0$, $Cx(t)+D(x(t)-r(t))=0$ and $I=0$ for all $x(t-r(t))=0$,so that system (2.1) admits a trivial solution.

As in [8],we have following definitions:

Definition 2.1. Let $p>0$, then the trivial solution to systems (2.1) is said to be pth moment globally uniformly exponentially stable if for any initial data $\xi\in L^b_{f_{t_0}}$ the solution $x(t;\xi)$ satisfies

$$\mathrm{E}(|x(t;\xi)|^p)\le C\mathrm{E}(\|\xi\|^p)e^{-\varepsilon(t-t_0)}, t\ge t_0 \tag{2.2}$$

where ε and C are positive constants independent of t_0 . It follows from (2.2) that

$$\lim_{t\to\infty}\sup\frac{1}{t}\log\mathrm{E}(|x(t;\xi)|^p)\le-\varepsilon. \tag{2.3}$$

The left-hand side of (2.3) is called the pth moment Lyapunov exponent for the solution.

Definition 2.2. Let $\ell^{1,2}$ denote the set of all functions from $[t_0-r,\infty)\times R^n$ to R^+ that are continuously differentiable in t and twice continuously differentiable in x . For each $V\in\ell^{1,2}$, define an operator from $R^+\times PL$ to R by

$$LV(t,\phi):=V_t(t,\phi(0))+V_x(t,\phi(0))f(t,\phi)$$
$$+\frac{1}{2}trace[g^T(t,\phi)V_{xx}(t,\phi(0))g(t,\phi)]$$

Consider the following impulsive stabilization of stochastic delay differential equations:

$$\begin{cases} dx(t)=f(t,x_t)dt+g(t,x_t)dW(t), t\neq t_k, t>t_0 \\ \Delta x(t)=Ix(t^-), t=t_k, \\ x_{t_0}=\xi, \end{cases}$$

We have the lemma:

Lemma. Let $\Lambda, p, c_1, c_2, \rho < 1, \delta$ and $\bar{\mu}$ be positive constants. Suppose that

(i) there exists a function $V \in \ell^{1,2}$ such that $c_1 |x|^p \le V(t,x) \le c_2 |x|^p$, for $(t,x) \in [t_0 - r, \infty) \times R^n$

(ii) there exists a nonnegative and piecewise continuous function $\mu : [t_0, \infty) \to R^+$, satisfying

$\int_t^{t+\delta} \mu(s) ds \le \bar{\mu} \delta$ for all $t \ge t_0$ .such that $\mathrm{E}(LV(t,\phi)) \le \mu(t) \mathrm{E}(V(t,\phi(0)))$

whenever $t \ge t_0$ and $\phi \in L^b_{F_f}$, are such that $\mathrm{E}(V(t+s,\phi(s))) \le q \mathrm{E}(V(t,\phi(0)))$, for all $s \in [-r, 0]$, where q is a constant such that $q \ge \exp(\wedge r + \wedge \delta + \bar{\mu} \delta)$,

(iii) there exist positive constants d_k , with $\prod_{k=1}^{\infty} d_k < \infty$, such that $\mathrm{E}(V(t_k, \phi(0) + I(t_k, \phi))) \le \rho d_k \mathrm{E}(V(t_k^-, \phi(0)))$ for $k \in \mathrm{Z}^+, \phi \in L^b_{F_f}$ and

(iv) $\sup_{k \in Z^+} \{t_k - t_{k-1}\} = \delta < -\dfrac{\ln(\rho)}{\wedge + \bar{\mu}}$

Then the trivial solution of system (2.1) is pth moment globally uniformly exponentially stable and its pth moment Lyapunov exponent is not greater than $-\wedge$.

In the following, we shall establish several criteria on impulsive stabilization of stochastic delay differential equations. Our result shows that even if the impulsive stabilization of stochastic delay differential equations are highly unstable, the impulsive control can successfully stabilize system(2.1).

III. Main results

Consider the impulsive stabilization of stochastic delay differential equations(2.1).

Theorem 3.1. Assume that there exist several positive constants $\mu, \Lambda, c_1, c_2, \delta$, $p \ge 2, \rho < 1$ such that

(i) that there exist two positive constants a, b such that $\|A(t)\| \le a, \|B(t)\| \le b$ for any $t \in [t_0, \infty)$

(ii) $|x(t - r(t))| \le |x(t)|$ for any single time-varying delay $r(t)$

(iii) $\mu \ge pa + pb + \dfrac{1}{2} p(p-1) \|C\|^2 + \dfrac{1}{2} p(p-1) \|D\|^2 + p(p-1) \|C\| \|D\|$

(iv) $\sup_{k \in Z^+} \{t_k - t_{k-1}\} = \delta < -\dfrac{\ln(\rho)}{\wedge + \mu}$

(v) there exists a positive constant $q > 1$ such that $q \ge \exp(\wedge r + \wedge \delta + \mu \delta)$

Then the trivial solution of system (2.1) is pth moment globally uniformly exponentially stable and its pth moment Lyapunov exponent is not greater than $-\wedge$.

Proof. Let $x(t) = x(t;\xi)$ be any solution of the system(2.1),given any initial data $\xi \in L^b_{f_{t_0}}$ .we assume that the initial data ξ is nontrivial so that $x(t)$ is not a trivial solution.

Let $v(t) = \mathrm{E}(V(t, x(t)))$, $V(x(t)) = |x|^p$ for $t \ge t_0 - r$, and $\tilde{\wedge} = \wedge - \eta$, where $\eta > 0$ being an arbitrary number such that $\tilde{\wedge} > 0$.

We can choose a constant $M \in (e^{(\wedge + \mu)\delta}, q e^{\wedge \delta})$ so that

$$\|v_{t_0}\| < M \|v_{t_0}\| e^{-(\wedge + \mu)\delta} < M \|v_{t_0}\| e^{-\wedge \delta} < q \|v_{t_0}\| \quad (1)$$

where $\|v_{t_0}\| = \max_{-r \le s \le 0} v(t_0 + s)$

We first prove that

$$v(t) \le M \|v_{t_0}\| e^{-\tilde{\wedge}(t_1 - t_0)}, \forall t \in [t_0, t_1) \quad (2)$$

To do this, we only need to prove a stronger claim:

$$v(t) \le M \|v_{t_0}\| e^{-\wedge \delta}, \forall t \in [t_0, t_1) \quad (3)$$

If (3) is not true, then for

$$v(t) \le \|v_{t_0}\| < M \|v_{t_0}\| e^{-\wedge \delta} \quad (4)$$

holds on $[t_0 - r, t_0]$.

Define $t^* = \inf\{t \in [t_0, t_1) : v(t) > M \|v_{t_0}\| e^{-\wedge \delta}\}$.

Then $t^* \in (t_0, t_1)$

$$v(t) \le v(t^*) = M \|v_{t_0}\| e^{-\wedge \delta}, \forall t \in [t_0, t^*] \quad (5)$$

In view of (4) .

Define $t_* = \sup\{t \in [t_0, t^*) : v(t) \le \|v_{t_0}\|\}$.

Then $t_* \in [t_0, t^*)$

$$v(t) \ge v(t_*) = \|v_{t_0}\|, \forall t \in [t_*, t^*] \quad (6)$$

By (4) (5) (6) ,we have ,for any $t \in [t_*, t^*]$ and $s \in [-r, 0]$

$$v(t+s) \le v(t^*) = M \|v_{t_0}\| e^{-\wedge \delta} < q \|v_{t_0}\| \le q v(t)$$

Then by lemma condition (ii)

$$\mathrm{E}(LV(t, x(t))) \le \mu(t) \mathrm{E}(V(t, x(t))), \ \forall t \in [t_*, t^*] \quad (7)$$

We get

$$LV(t,x(t)) = p|x|^{p-1}[A(t)x(t) + B(t)x(t-r(t))] + p(p-1)|x|^{p-2}[C(t) + D(t-r(t))]\} + \frac{1}{2} trace\{[Cx(t) + Dx(t-r(t))]^T$$

By condition (i) (ii) we get

$$LV(t, x(t)) \le p|x|^{p-1} a |x(t)| + p|x|^{p-1} b |x(t - r(t))|$$

$$+\frac{1}{2}p(p-1)|x|^{p-2}\{[Cx(t)+Dx(t-r(t))]^T[Cx(t)+Dx(t-r(t))]\}$$

$$\le pa|x|^p+pb|x|^{p-1}|x(t-r(t))|+\frac{1}{2}p(p-1)|x|^{p-2}\sum[(C_{li}x_i(t)+D_{li}x_i(t-r(t))]^2$$

$$\le pa|x|^p+pb|x|^{p-1}|x(t-r(t))|$$

$$+\frac{1}{2}p(p-1)|x|^{p-2}\sum\{[C_{li}x_i(t)]^2+2[C_{li}x_i(t)][D_{li}x_i(t-r(t))]+[D_{li}x_i(t-r(t))]^2\}$$

$$\le pa|x|^p+pb|x|^{p-1}|x(t-r(t))|+\frac{1}{2}p(p-1)|x|^{p-2}\|C\|^2|x|^2+$$

$$\frac{1}{2}p(p-1)|x|^{p-2}\|D\|^2|x(t-r(t))|^2+p(p-1)|x|^{p-2}\|C\|\|D\||x||x(t-r(t))|$$

$$\le pa|x|^p+pb|x|^p+\frac{1}{2}p(p-1)\|C\|^2|x|^p+\frac{1}{2}p(p-1)\|D\|^2|x|^p+p(p-1)\|C\|\|D\||x|^p$$

$$=(pa+pb+\frac{1}{2}p(p-1)\|C\|^2+\frac{1}{2}p(p-1)\|D\|^2+p(p-1)\|C\|\|D\|)V(x(t))$$

By condition （iii）,we get

$$\mathrm{E}(LV(t,x(t)))\le\mu\mathrm{E}V(x(t)) \qquad (8)$$

Applying Itô's formula on $[t_*,t^*]$ and by （8）,we get

$$e^{\int_{t_0}^{t^*}\mu d\tau}v(t^*)-e^{\int_{t_0}^{t_*}\mu d\tau}v(t_*)$$

$$=\int_{t_*}^{t^*}e^{\int_{t_0}^{s}\mu d\tau}[\mathrm{E}(LV(s,x(s)))-\mathrm{E}V(x(s))]ds\le 0$$

which shows $v(t^*)\le v(t_*)e^{\int_{t_*}^{t^*}\mu ds}\le v(t_*)e^{\mu\delta}$ (9)

which is a contradiction. Hence (4) holds and (2) is true.

Now we assume that for any

$$t\in[t_{k-1},t_k),\ v(t)\le M_k\|v_{t_0}\|e^{-\tilde{\wedge}(t_k-t_0)} \qquad (10)$$

Where $k\le m$， $k,m\in\mathrm{Z}^+$， define $M_k:M_1=M$， $M_k=M\prod_{1\le l\le k-1}d_l$ when $k\ge 2$

Next, we shall prove that

$$v(t)\le M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)},\forall t\in[t_m,t_{m+1}) \qquad (11)$$

If (11) is not true,for condition （iv）,lemma condition （iii）and(10),we get

$$v(t_m)\le\rho d_m v(t_m^-)$$

$$\le\rho d_m M_m\|v_{t_0}\|e^{-\tilde{\wedge}(t_m-t_0)}=\rho M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_m-t_0)}$$

$$<e^{-\delta(\wedge+\mu)}M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_m-t_0)}<e^{-\delta\mu}M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)} \qquad (12)$$

Define

$$\bar{t}=\inf\{t\in[t_m,t_{m+1}):v(t)>M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)}\}.$$

Then $\bar{t}\in(t_m,t_{m+1})$

$$v(t)<M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)},\quad t\in[t_m,\bar{t}) \qquad (13)$$

Define $\underline{t}=\sup\{t\in[t_m,\bar{t}):v(t)\le e^{-\mu\delta}M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)}\}$.

Then $\underline{t}\in(t_m,\bar{t})$

$$v(t)\ge v(\underline{t})=e^{-\mu\delta}M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t_{m+1}-t_0)}=e^{-\mu\delta}v(\bar{t}),\quad t\in(\underline{t},\bar{t}) \qquad (14)$$

Now for $t\in[\underline{t},\bar{t}]$ and $s\in[-r,0]$, from(10) and (13), and the fact that $q\ge\exp(\wedge r+\wedge\delta+\mu\delta)$ and $t+s\in[t_{m-1},\bar{t}]$,we get

$$v(t+s)\le M_{m+1}\|v_{t_0}\|e^{-\tilde{\wedge}(t+s-t_0)}\le e^{\tilde{\wedge}r+\tilde{\wedge}\delta+\mu\delta}v(\underline{t})\le qv(t)$$

Similar to the argument on $[t_*,t^*]$, an application of Itô's formula on $[\underline{t},\bar{t}]$ will lead to $v(\bar{t})\le v(\underline{t})e^{\mu\delta}$, which would contradict(14). Therefore, claim(11)must be true .

From the definition of M_m ,we have

$$v(t)\le M\prod_{\{k:t_0<t_k\le t\}}d_k\|v_{t_0}\|e^{-\tilde{\wedge}(t-t_0)},\forall t\ge t_0$$

By condition（i）,lemma condition （iii）and the arbitrary number $\eta>0$， we can get

$$\mathrm{E}(|x(t)|^p)\le M\tilde{d}\frac{c_2}{c_1}\mathrm{E}(\|\xi\|^p)e^{-\wedge(t-t_0)},\forall t\ge t_0 \quad ,$$

$\tilde{d}=\prod_{k=1}^{\infty}d_k<\infty$。

which implies that the trivial solution of system (2.1) is pth moment globally uniformly exponentially stable and its pth moment Lyapunov exponent is not greater than $-\wedge$.

Now consider the linear impulsive stochastic delay system：

$$\begin{cases}dx(t)=[Ax(t)+Bx(t-r(t))]dt+[Cx(t)+Dx(t-r(t))]dW(t),t\ne t_k,t>t_0\\ \Delta x(t)=Ix(t^-),t=t_k,\end{cases}$$

（3.1）

where A,B,C,D are $n\times n$ matrices , I is an identity matrix , W is a one-dimensional standard Wiener process. A single time-varying delay is given by $r(t)$, which is continuous on $[t_0,\infty)$ and satisfies $0\le r(t)\le r$. The initial data is omitted, but it is assumed to be in $\xi\in L^b_{f_{t_0}}$.

Corollary 3.1. If there exist constants $\Lambda>0,p\ge 2,\rho<1$ such that $q\ge\exp(\wedge r+\wedge\delta+\mu\delta)$, where

$$\mu=p\|A\|+p\|B\|+\frac{1}{2}p(p-1)\|C\|^2$$

$$+\frac{1}{2}p(p-1)\|D\|^2+p(p-1)\|C\|\|D\|$$

then the trivial solution of system （3.1） is pth moment globally uniformly exponentially stable, with its Lyapunov exponent not greater than $-\wedge$.

Proof. The conclusions follow from Theorem 3.1 by considering $V(x)=|x|^p$.

IV. AN EXAMPLE

Example 4.1. Consider the linear impulsive stochastic delay system

$$\begin{cases} dx(t)=[Ax(t)+Bx(t-\frac{1}{3}e^{-t})]dt+[Cx(t)+Dx(t-\frac{1}{3}e^{-t})]dW(t), t\neq t_k, t>t_0 \\ \Delta x(t)=Ix(t^{-}), t=t_k, \end{cases}$$

where I is an identity matrix , $t_k = 0.05$, time delays are $1/4$, $1/5$, $1/6$,

$$A=\begin{pmatrix} -0.24 & 0.15 & 0.35 \\ 0.09 & -0.35 & -0.65 \\ 0.45 & -0.38 & 0.26 \end{pmatrix}, \quad B=\begin{pmatrix} 0.34 & 0.31 & 0.08 \\ -0.28 & 0.25 & 0.38 \\ -0.67 & 0.16 & 0.02 \end{pmatrix},$$

$$C=\begin{pmatrix} 0.35 & 0.57 & 0.42 \\ -0.54 & 0.06 & 0.17 \\ -0.15 & 0.24 & 0.22 \end{pmatrix}, \quad D=\begin{pmatrix} 0.33 & 0.34 & 0.21 \\ -0.27 & 0.28 & 0.16 \\ -0.28 & 0.83 & 0.72 \end{pmatrix}$$

Choosing $p=2, \wedge=0.6, \delta=0.01$ then μ and q in Corollary 3.1 can be computed to be $\mu = 7.5734$, $q=\exp(\wedge r + \wedge\delta + \mu\delta) = 1.3254$. Numerical simulations for this example are shown in Fig.1 and Fig.2. Fig.1 shows that system response without impulses.Fig.2 shows that impulsive stabilization of system.It is clearly demonstrated that impulses can successfully stabilize an otherwise unstable stochastic delay system..

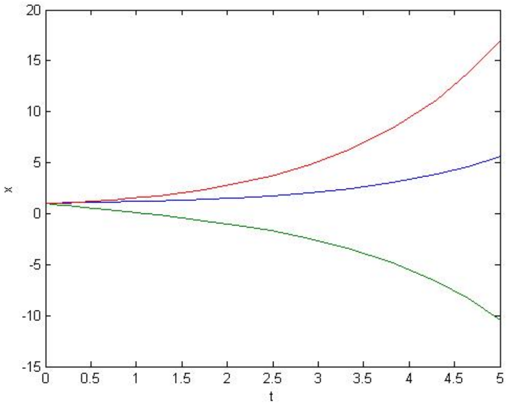

Fig 1. System without impulses.

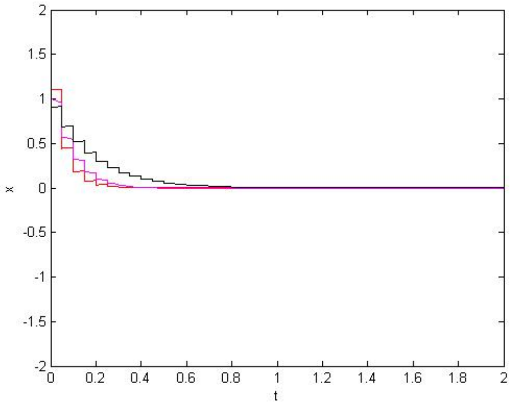

Fig. 2. Impulsively stabilized system.

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